The Stokes flow problem for a class of axially symmetric bodies

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The Stokes flow problem is concerned with fluid motion about an obstacle when the motion is such that inertial effects can be neglected. This problem is considered here for the case in which the obstacle (or configuration of obstacles) has an axis of symmetry, and the flow at distant points is uniform and parallel to this axis. The differential equation for the stream function ψ then assumes the form $L^2_{-1}\psi = 0$, where L_{-1} is the operator which occurs in axially symmetric flows of the incompressible ideal fluid. This is a particular case of the fundamental operator of A. Weinstein's generalized axially symmetric potential theory. Using the results of this theory and theorems regarding representations of the solutions of repeated operator equations, the authors (1) give a general expression for the drag of an axially symmetric configuration in Stokes flow, and (2) indicate a procedure for the determination of the stream function. The stream function is found for the particular case of the lens-shaped body.

Explicit calculation of the drag is difficult for the general lens, without recourse to numerical procedures, but is relatively easy in the case of the hemispherical cup. As examples illustrative of their procedures, the authors briefly consider three Stokes flow problems whose solutions have been given previously.

1. Introduction

The determination of the isothermal flow of an incompressible, viscous fluid about an impermeable body immersed therein requires the solution of the Navier–Stokes equations and the equation of continuity subject to the condition that the velocity of flow coincide with that of the external boundary of the body at each of its points. The non-linearity of the Navier–Stokes equations renders the solution of this problem extremely difficult, and only a relatively small number of exact solutions of rather specialized character are known (Dryden *et al.* 1932; Lamb 1932). In many instances, however, the flow is such that plausible simplifying assumptions regarding its character (in certain regions, at least) can be made which result in a less refractory mathematical problem. A large part of the theory of viscous flows consists in the discussion of problems obtained in this manner.

The oldest problem of this type is the so-called 'Stokes flow' problem. It is defined by the assumption that inertial effects are negligible in comparison with those of viscosity, or, more precisely, that the Reynolds number R of the flow is very small. This situation obtains when either the characteristic flow velocity or body dimension (or both) appearing in R is suitably small or the kinematic viscosity is large. Sir George Stokes (1850), in the course of treating the steady motion of a sphere in a viscous liquid, appears to have been the first to omit the inertial terms in the equations of motion. Oberbeck (1876) solved the Stokes flow problem for an ellipsoid immersed in a uniform flow, and also treated the special cases of a circular disk broadside and edgewise to the undisturbed flow. More recently, Ray (1936), by a more direct method, has obtained the solution of this circular disk problem in a different form. His solution yields the same drag as was obtained by Oberbeck. The problem of a sphere moving slowly near a plane wall was solved by Lorentz (1897), while that of a sphere falling along the axis of a vertical tube has been discussed by Ladenburg (1907).

A number of two-dimensional flows of Stokes type have also been treated. Berry & Swain (1923) have considered the elliptic cylinder immersed in an infinite body of fluid, as well as the special cases of the circular cylinder and the infinitely thin flat plate athwart and edgewise to the flow. The speed of flow becomes logarithmically infinite at points infinitely far from any of these bodies. Dean has devoted a series of papers (1944; others are referenced there) to the Stokes flow past protuberances of various shapes in otherwise flat walls, the fluid filling the half-space lying on that side of the wall which bears the protuberance. Here again the speed becomes infinite at an infinite distance from the wall. Dean (1936) has also considered the Stokes flow problem for a fluid which emerges uniformly from an aperture in a plane wall into a semi-infinite body of fluid. In this case the speed of flow vanishes at infinity. Green (1944) has found a Stokes flow for the region bounded by converging plane walls and for the region lying between two branches of a hyperbola. Again the speed vanishes at infinity. Further discussion and bibliography on Stokes flows may be found in Dryden *et al.* (1932) and Lamb (1932).

It seems reasonable to doubt the uniqueness of the Berry–Swain solutions, since it appears not unlikely that there may exist a Stokes flow about the cylinder which is uniform at infinity, in view of the Stokes and Oberbeck results for the sphere and ellipsoid. It has been generally accepted, however, since the time of Stokes, that such is not the case, Stokes himself giving a physical argument for this apparent anomaly (1850). The question has recently been settled once for all by Finn & Noll (1957), who have given a careful proof of the fact that the only Stokes flow uniform at infinity about a cylinder whose boundary is composed of a finite number of piecewise smooth non-intersecting simple closed curves lying in the finite plane is the state of rest. The boundaries in the other two-dimensional problems mentioned above do not lie entirely in the finite plane, and so the Finn–Noll theorem does not apply.

In the case of three-dimensional flow, the prescription of a uniform velocity at infinity in a Stokes flow gives a well-set problem, for Finn & Noll (1957) have demonstrated the uniqueness of such a flow about a body whose surface is composed of a finite number of piecewise smooth non-intersecting simple closed surfaces. See also Odqvist (1930).

In this paper we propose to consider the solution of the Stokes flow problem for axially symmetric bodies with the aid of the generalized axially symmetric potential theory initiated and developed by Weinstein (1948, 1955) and his co-workers (Huber 1953; Hyman 1954; Payne 1958). After the formulation of the problem and a general discussion of the techniques employed in its solution, we restrict our discussion to simply connected regions (in a meridional plane). We first obtain a general expression for the drag of the simply connected axially symmetric body, and then solve the Stokes flow problem for the general lens-shaped body. Some interesting special cases of this body are discussed. As examples of the method, some previously solved problems are discussed briefly. Finally, a table of the drag of various bodies in Stokes flow is given.

2. Statement of the Stokes flow problem for axially symmetric bodies

Consider a collection of n bodies which are individually axially symmetric and which are so arranged that the same is true of the aggregate of bodies. Let these be immersed at rest in a uniform flow of a viscous, incompressible fluid which fills three-dimensional space, and let the axis of symmetry be parallel to the direction of the uniform flow (referred to as the free stream direction). We refer the flow to cylindrical co-ordinates (x, r, θ). The x-axis is chosen to lie along the axis

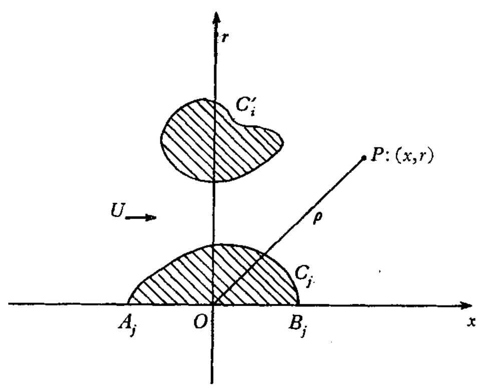

FIGURE 1. The general configuration.

of symmetry of the body with its positive direction the same as that of the free stream, r is radial distance from this axis, and θ is an azimuthal angle defining meridional planes through the x-axis. In view of the body-flow–co-ordinate configuration the co-ordinate θ will play no role in our analysis, and we may restrict attention to any single meridional half-plane, as shown in figure 1, so that the range of co-ordinates is $-\infty \leqslant x \leqslant \infty$, $0 \leqslant r \leqslant \infty$, and $0 \leqslant \theta \leqslant 2\pi$. A meridional section of the bodies consists of regions bounded by two types of curves: (i) closed contours C_i', $i = 1, 2, \ldots, l$, each of which lies entirely above the x-axis, and (ii) arcs C_j, $j = 1, 2, \ldots, k$ $(k+l = n)$, which have termini A_j and B_j, but no other points, lying on the x-axis (figure 1). The region of flow D is that portion of the meridional half-plane $r \geqslant 0$ which is exterior to the regions enclosed by the C_i' and the $C_j + A_jB_j$. The C_j and C_i' are assumed to be made up of a finite number of smooth arcs, joined end to end. For brevity, we shall refer to a meridional section of a body as 'the body', and the bounding curve C_i or C_i' as its profile. The complete set of profiles constitutes the boundary of the set of bodies and is denoted by C. The complete boundary of the *flow region*, however, consists of C together with those segments of the x-axis exterior to the set of $A_j B_j$, $j = 1, 2, \ldots, k$. The velocity of flow at a generic point (x, r) of D we denote by $\mathbf{u}(x,r) = (u_x(x,r), u_r(x,r))$, and the undisturbed (free stream) velocity by $\mathbf{u}_\infty = (U, 0)$, where $U > 0$ is a constant. We assume that

$$\lim_{\rho \to \infty} \mathbf{u}(x,r) = \mathbf{u}_\infty, \tag{2.1}$$

where

$$\rho = (x^2 + r^2)^{\frac{1}{2}}. \tag{2.2}$$

Since the fluid is incompressible and the flow steady, the continuity equation becomes simply

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D.$$

The axial symmetry of the flow permits the introduction of a stream function ψ. Accordingly, we write

$$u_x = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial x}, \tag{2.3}$$

and the continuity equation is then of necessity satisfied.

The vorticity vector $\boldsymbol{\zeta} = \operatorname{curl} \mathbf{u}$ has at each point of the meridional plane the form $(0, 0, \zeta)$ and it is thus sufficient to deal only with the scalar ζ. It is seen at once (Milne-Thomson 1950, p. 494) that

$$\zeta = \frac{\partial u_r}{\partial x} - \frac{\partial u_x}{\partial r}, \tag{2.4}$$

and if (2.3) are inserted in this we obtain

$$\zeta = -\frac{1}{r} L_{-1} \psi, \tag{2.5}$$

where

$$L_{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{2.6}$$

The reason for the departure from the usual notation (Milne-Thomson 1950, p. 494) for this operator will be clear later.

By a well-known procedure (Milne-Thomson 1950, pp. 507–9) the Navier–Stokes equations can be converted into the following equation for ζ in D:

$$\frac{\partial}{\partial r}(\zeta u_r) + \frac{\partial}{\partial x}(\zeta u_x) + \frac{\nu}{r} L_{-1}(r\zeta) = 0, \tag{2.7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. Substitution from (2.3) and (2.5) then permits us to write this in the form

$$\frac{\partial[\psi, (1/r^2) L_{-1}\psi]}{\partial(r, x)} - \nu L_{-1}^2 \psi = 0. \tag{2.8}$$

The Stokes assumption asserts that the first term of this is negligible in comparison with the second, and we thus obtain for axially symmetric Stokes flow the equation

$$L_{-1}^2 \psi = 0 \tag{2.9}$$

to be satisfied in D.

The kinematic condition of vanishing normal and tangential components of $\mathbf{u}$ at the boundary of an impermeable body at rest in a viscous fluid leads to the conditions

$$\psi = k, \text{ a constant,} \tag{2.10a}$$

$$\frac{\partial \psi}{\partial n} = 0, \tag{2.10b}$$

to be satisfied on each C_i and C'_j. We take $\mathbf{n}$ to be the unit normal on C directed into the fluid. In general, k will be different on different C'_j, although it will have the same value on all C_i.

Finally, it is easy to see from (2.1) and (2.3) that ψ must satisfy the condition

$$\lim_{\rho\to\infty} \psi(x,r) = \tfrac{1}{2}r^2 U \tag{2.11}$$

(apart from an arbitrary additive constant which we have taken to be zero).

In summary: the solution $\psi(x,r)$ of (2.9) subject to the conditions (2.10) and (2.11) solves the problem of Stokes flow for an axially symmetric body immersed in an infinite fluid whose velocity at infinity is parallel to the axis of the body.

3. Representation of the solution of the flow equation

We follow the usual procedure in solving boundary value problems: we obtain solutions of the differential equation (2.9) which seem suitable for our purpose, and construct therefrom by the well-known linear operations one which also satisfies the boundary conditions (2.10)–(2.11). The writers believe the problem considered here to have some novelty, however, inasmuch as the differential equation (2.9) is one which has received little attention and also because the solution given here constitutes an application of powerful methods recently developed by Weinstein (1948, 1955), Payne (1958), and other workers (as indicated below) in generalized axially symmetric potential theory.

Let $\psi^k(x,r)$ denote any solution of

$$L_k(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{k}{r}\frac{\partial v}{\partial r} = 0 \tag{3.1}$$

$(-\infty \leqslant k \leqslant \infty)$ in D which is regular on the x-axis outside of C. We refer to ψ^k as a generalized axially symmetric potential function. The behaviour of such functions in the neighbourhood of the x-axis has been discussed for all real k by Hyman (1954) and Huber (1953). It can be checked by direct substitution that the following functions are solutions of equation (2.9) for the stream function:

$$(a)\ r^2\psi^3, \quad (b)\ xr^2\psi^3, \quad (c)\ \rho^2 r^2\psi^3, \quad (d)\ r^2\psi^1, \quad (e)\ r^4\psi^5. \tag{3.2}$$

Moreover, Payne (1958) has shown that in certain regions any solution of (2.9) can be represented as a linear combination of any two of the expressions (3.2). It may be surmised (and this will be verified in the examples we shall consider) that the optimal combination to be chosen as a solution for a specific problem will depend on the geometry of the boundary C on which (2.10) and (2.11) must be satisfied. A combination suitable for one problem may be completely intractable for another. In fact, we choose quite different combinations in discussing the flow about a spheroid from those used in the case of the lens-shaped region or separated spheres.

In the remainder of this paper we shall restrict our attention to the case in which only profiles of the type C_j occur. The connected curve formed by the $C_j, j = 1, 2, \ldots, k$, and those segments of the x-axis exterior to the $A_j B_j$ (figure 2)

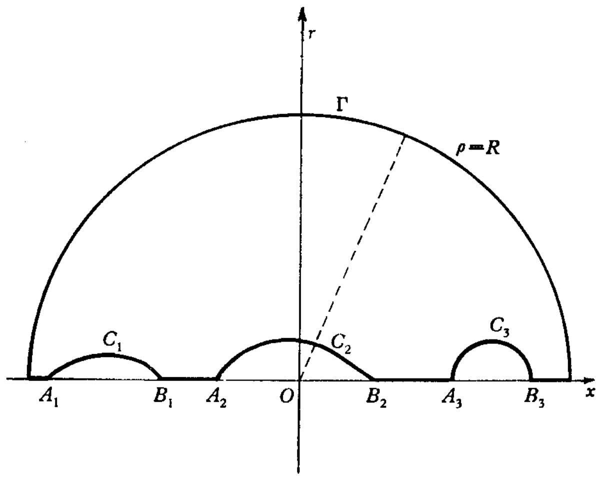

FIGURE 2. Configuration for simply connected flow field.

then form a streamline of the flow, and, in fact, constitute the boundary of D. Thus D is simply connected for this configuration. Since $\psi = 0$ on the x-axis at infinity, it must remain so along the entire streamline above, i.e. on C and the x-axis exterior to the bodies. Condition (2.10b) holds on each C_j and thus on C. Conditions (2.10) revised for the case under consideration are thus

$$\psi = 0, \tag{3.3a}$$

$$\frac{\partial \psi}{\partial n} = 0, \tag{3.3b}$$

on C.

If bodies of type C_i' occur, then D is multiply connected, and (3.3a) no longer holds.

It is convenient to write the stream function ψ in the form

$$\psi = \tfrac{1}{2}Ur^2 - \psi_1 \tag{3.4}$$

and to formulate the problem in terms of ψ_1 rather than ψ. It is clear that ψ_1 satisfies the same differential equation (2.9) as ψ. In addition, it must give rise to a vanishing velocity at infinity, and fulfil the conditions

$$\psi_1 = \tfrac{1}{2}Ur^2, \tag{3.5a}$$

$$\frac{\partial \psi_1}{\partial n} = Ur\frac{\partial r}{\partial n}, \tag{3.5b}$$

on C.

4. Drag on the axially symmetric body

The drag on an axially symmetric body at rest in the flow, expressed in cylindrical co-ordinates, can be shown to be

$$P = \frac{2\pi\mu}{U}\iint_D r\zeta^2\,dr\,dx = \frac{2\pi\mu}{U}\iint_D \frac{1}{r}[L_{-1}\psi_1]^2\,dr\,dx \tag{4.1}$$

(see Milne-Thomson 1950, §19.21).

We next show that this integral for the drag may be replaced by the expression

$$\frac{P}{8\pi\mu} = \lim_{\rho\to\infty}\frac{\rho\psi_1}{r^2}. \tag{4.2}$$

To this end we use the identity

$$\iint_{D^*}\frac{1}{r}[uL_{-1}v - vL_{-1}u]\,dx\,dr = \int_{C^*}\frac{1}{r}\left[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right]ds, \tag{4.3}$$

which can be obtained from the divergence theorem by a procedure resembling that used in deriving the symmetric form of Green's theorem (see Weinstein (1948), equation (20)). The functions u and v, the region D^* and its boundary C^* are subject to the hypotheses of that theorem, and $\mathbf{n}$ is a unit normal to C^* exterior to D^*.

We now set $u = L_{-1}\psi_1$, $v = \psi_1$, and apply (4.3) to the region D^* whose complete boundary C^* consists of (i) the arcs C_j, (ii) a semicircle Γ defined by $\rho = R$, $r \geqslant 0$, where R is arbitrary save that all profiles C_j must lie within $\rho < R$, and (iii) those segments of the x-axis which lie between the termini of Γ and are exterior to the bodies. Since $L_{-1}^2\psi_1 = 0$, we see at once that we then have

$$\iint_{D^*}\frac{1}{r}[L_{-1}(\psi_1)]^2\,dr\,dx = \int_{C^*}\frac{1}{r}\left\{\psi_1\frac{\partial}{\partial n}L_{-1}(\psi_1) - L_{-1}(\psi_1)\frac{\partial\psi_1}{\partial n}\right\}ds. \tag{4.4}$$

Since ψ_1 is a solution of (2.9) it follows from the significance of the notation that $L_{-1}\psi_1 = \psi^{-1}$. We now make the plausible assumption that at all points of C on the x-axis the velocity gradients are bounded. It then follows from equation (2.5) and the results of Huber (1953) and Hyman (1954) that $L_{-1}\psi_1$ is $O(r^2)$ as $r\to 0$ along the x-axis outside the body. Moreover, it is apparent from either the decomposition formula of Weinstein (1955) or those of Payne (1958) (which are valid in the neighbourhood of $r = 0$) and the results of Huber and Hyman that if ψ_1 vanishes at ∞ on the x-axis and is analytic in a neighbourhood of a segment of the x-axis it must vanish as r^2 there. From these remarks it follows that the integral on the right-hand side of (4.4) arising from that portion of C^* which lies along the x-axis vanishes. We thus have

$$\iint_{D^*}\frac{1}{r}[L_{-1}(\psi_1)]^2\,dr\,dx = \int_\Gamma\frac{1}{r}\left\{\psi_1\frac{\partial}{\partial n}L_{-1}(\psi_1) - L_{-1}(\psi_1)\frac{\partial\psi_1}{\partial n}\right\}ds$$
$$+\int_{C^*-\Gamma}\frac{1}{r}\left\{\psi_1\frac{\partial}{\partial n}L_{-1}(\psi_1) - L_{-1}(\psi_1)\frac{\partial\psi_1}{\partial n}\right\}ds, \tag{4.5}$$

which may be rewritten as

$$\iint_{D^*} \frac{1}{r} [L_{-1}(\psi_1)]^2 \, dr \, dx = \int_\Gamma \frac{1}{r} \left\{ (\psi_1 - \tfrac{1}{2}Ur^2) \frac{\partial}{\partial n} L_{-1}(\psi_1) - L_{-1}(\psi_1) \frac{\partial}{\partial n} (\psi_1 - \tfrac{1}{2}Ur^2) \right\} ds$$
$$+ U \int_{C^*} \frac{1}{r} \left\{ \frac{r^2}{2} \frac{\partial}{\partial n} L_{-1} \psi_1 - r \frac{\partial r}{\partial n} L_{-1}(\psi_1) \right\} ds. \quad (4.6)$$

Since $L_{-1}(r^2) = 0$ and ψ_1 is a solution of (2.9) we may use (4.3) to show that the integral over C^* above vanishes, and we obtain the important result that

$$\iint_{D^*} \frac{1}{r} [L_{-1}(\psi_1)]^2 \, dr \, dx = \int_\Gamma \frac{1}{r} \left\{ (\psi_1 - \tfrac{1}{2}Ur^2) \frac{\partial}{\partial n} L_{-1}(\psi_1) - L_{-1}(\psi_1) \frac{\partial}{\partial n} (\psi_1 - \tfrac{1}{2}Ur^2) \right\} ds. \quad (4.7)$$

From the decompositions (3.2) given by Payne it follows that we may write

$$\psi_1 = r^2(\psi^1 + \psi^3) \quad (4.8)$$

in any region sufficiently far removed from the body. Since ψ^1 and ψ^3 may, however, be regarded as axially symmetric harmonic functions in spaces of 3 and 5 dimensions, respectively, (4.8) yields the representation

$$\psi_1 = r^2 \left[\frac{K}{\rho} + O\left(\frac{1}{\rho^2}\right) \right] \quad (4.9)$$

(K a constant) in the neighbourhood of $\rho = \infty$. Further, we may conclude from this that

$$L_{-1} \psi_1 = -\frac{2r^2 K}{\rho^3} + O\left(\frac{1}{\rho^2}\right) \quad (4.10)$$

in the neighbourhood of $\rho = \infty$.

With this information we return to (4.7) and let $R \to \infty$ in that equation. This yields, since $D^* \to D$ as $R \to \infty$,

$$\frac{PU}{2\pi\mu} = \lim_{R \to \infty} \int_0^\pi 3KU \frac{r^3}{R^3} \left[1 + O\left(\frac{1}{R}\right) \right] d\theta = 3KU \int_0^\pi \sin^3\theta \, d\theta,$$

or

$$P = 8\pi\mu K. \quad (4.11)$$

In view of (4.9) $K = \lim_{\rho \to \infty} \rho\psi_1/r^2$, and we have therefore established (4.2). We now turn to the consideration of a specific flow configuration.

5. The flow about a lens-shaped body

In order to compute the flow about a lens-shaped body we introduce peripolar co-ordinates (Hobson 1931, pp. 422, 451) (ξ, η) in the plane of the (x, r) co-ordinates through the transformation

$$x = \frac{b \sin \xi}{\cosh \eta - \cos \xi}, \quad r = \frac{b \sinh \eta}{\cosh \eta - \cos \xi}. \quad (5.1)$$

In effect, this is a dipolar system of co-ordinates in which the family of co-axial circles nest about $(0, \pm b)$. These circles are the curves $\eta = \text{const.}$, while the curves $\xi = \text{const.}$ are arcs of circles joining $(0, \pm b)$ (i.e. arcs of the orthogonal trajectories of the circles $\eta = \text{const.}$). As before, we consider only the half-plane $r \geqslant 0$.

The region bounded by the surface of revolution obtained by revolving two arcs $\xi = \xi_1$, and $\xi = \xi_2$ about the x-axis is referred to as a lens-shaped region (figure 3). A lens-shaped body is thus one which occupies a lens-shaped region. The interior of the region is defined by

$$0 < \xi_1 < \xi < \xi_2 < 2\pi \quad (0 < \eta < \infty) \tag{5.2}$$

and the exterior is then given by

$$\xi_2 < \xi < \xi_1 + 2\pi \quad (0 < \eta < \infty). \tag{5.3}$$

Accordingly, our problem is to find a solution of (2.9) which satisfies the conditions

$$\left.\begin{aligned} \psi &= 0, &(5.4a)\\ \frac{\partial\psi}{\partial\xi} &= 0, &(5.4b)\end{aligned}\right\} \quad \text{on} \quad \xi = \xi_1, \xi_2.$$

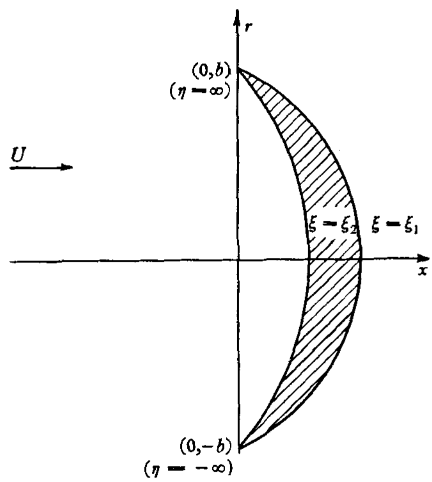

FIGURE 3. The lens-shaped body.

Let ξ_0 be a value of ξ such that

$$\xi_1 < \xi_0 < \xi_2. \tag{5.5}$$

We choose for ψ a representation of the form

$$\psi = \frac{Ur^2}{2}\left\{1 - \frac{(s-t)^{\frac{1}{2}}}{(s-\cos(\xi-\xi_0))^{\frac{1}{2}}} - (s-t)^{\frac{1}{2}}\int_0^\infty F(\alpha,\xi)\, K_\alpha^{(1)}(s)\, d\alpha\right\}, \tag{5.6}$$

where

$$F(\alpha,\xi) = \cos\xi[A(\alpha)\cosh\alpha\xi + B(\alpha)\sinh\alpha\xi] + \sin\xi[C(\alpha)\cosh\alpha\xi + D(\alpha)\sinh\alpha\xi]. \tag{5.7}$$

The functions $A(\alpha), \ldots, D(\alpha)$ are to be determined by the boundary conditions, and

$$s = \cosh\eta, \quad t = \cos\xi. \tag{5.8}$$

After Hobson (1931, pp. 444–53), we denote the Legendre function of complex degree $P_{i\alpha-\frac{1}{2}}(s)$, usually referred to as a conal function, by

$$K_\alpha(s) \equiv P_{i\alpha-\frac{1}{2}}(s). \tag{5.9}$$

The notation

$$K_\alpha^{(n)}(s) \equiv \frac{d^n}{ds^n} P_{i\alpha-\frac{1}{2}}(s) \quad (n = 1, 2, \ldots) \tag{5.10}$$

will also be used.

The condition (5.4a) leads to the equations

$$\int_0^\infty F(\alpha, 2\pi+\xi_1)\, K_\alpha^{(1)}(s)\, d\alpha = [s-t_1]^{-\frac{1}{2}} - [s-\cos(\xi_1-\xi_0)]^{-\frac{1}{2}}, \tag{5.11a}$$

$$\int_0^\infty F(\alpha, \xi_2)\, K_\alpha^{(1)}(s)\, d\alpha = [s-t_2]^{-\frac{1}{2}} - [s-\cos(\xi_2-\xi_0)]^{-\frac{1}{2}}, \tag{5.11b}$$

where $t_i = \cos\xi_i$, $i = 1, 2$.

From the integral definition of the conal function $K_\alpha(s)$ given in Hobson (1931, p. 451), and the subsequent discussion given there, it can be shown that the expansion

$$(s-t)^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty \frac{\cosh\alpha(\xi-\pi)}{\cosh\alpha\pi} K_\alpha(s)\, d\alpha \tag{5.12}$$

is valid for $0 < \xi < 2\pi$. Further, by differentiation and a permissible interchange of order of this with the indicated integration, we obtain

$$-\tfrac{1}{2}(s-t)^{-\frac{3}{2}} = -\sqrt{2}\int_0^\infty \frac{\cosh\alpha(\xi-\pi)}{\cosh\alpha\pi} K_\alpha^{(1)}(s)\, d\alpha. \tag{5.13}$$

Both members of this are now multiplied by $\sin\xi$ and the result integrated from $\xi = \xi_1$ to $\xi = 2\pi-(\xi_0-\xi_1)$. We obtain the representation

$$\begin{aligned}[s-\cos(\xi_1-\xi_0)]^{-\frac{1}{2}} - [s-t_1]^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty &\frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi} \\ &\times\{\cos\xi_1\cosh\alpha(\xi_1-\pi) - \cos(\xi_0-\xi_1)\cosh\alpha(\pi-\xi_0+\xi_1) \\ &-\alpha[\sin\xi_1\sinh\alpha(\xi_1-\pi) + \sin(\xi_0-\xi_1)\sinh\alpha(\pi-\xi_0+\xi_1)]\}\, d\alpha\end{aligned} \tag{5.14}$$

for $0 < \xi_1 < \xi_0$. By an entirely similar procedure we find that

$$\begin{aligned}[s-t_2]^{-\frac{1}{2}} - [s-\cos(\xi_2-\xi_0)]^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty &\frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi} \\ &\times\{\alpha\sin\xi_2\sinh\alpha(\xi_2-\pi) - \alpha\sin(\xi_2-\xi_0)\sinh\alpha(\xi_2-\xi_0-\pi) \\ &-\cos\xi_2\cosh\alpha(\xi_2-\pi) + \cos(\xi_2-\xi_0)\cosh\alpha(\xi_2-\xi_0-\pi)\}\, d\alpha\end{aligned} \tag{5.15}$$

for $\xi_0 < \xi_2 < 2\pi$. We may now replace the right-hand members of (5.11a) and (5.11b), respectively, by (5.14) and (5.15), and we are then led to the equations

$$\begin{aligned}F(\alpha, \xi_1+2\pi) = \frac{\sqrt{2}}{(\alpha^2+1)\cosh\alpha\pi}&\{\alpha\sin(\xi_0-\xi_1)\sinh\alpha(\pi-\xi_0+\xi_1) \\ &+\alpha\sin\xi_1\sinh\alpha(\xi_1-\pi) + \cos(\xi_0-\xi_1)\cosh\alpha(\pi-\xi_0+\xi_1) \\ &-\cos\xi_1\cosh\alpha(\xi_1-\pi)\}\end{aligned} \tag{5.16}$$

for $0 < \xi_1 < \xi_0$, and

$$\begin{aligned}F(\alpha, \xi_2) = \frac{\sqrt{2}}{(\alpha^2+1)\cosh\alpha\pi}&\{\alpha\sin\xi_2\sinh\alpha(\xi_2-\pi) - \alpha\sin(\xi_2-\xi_0)\sinh\alpha(\xi_2-\xi_0-\pi) \\ &-\cos\xi_2\cosh\alpha(\xi_2-\pi) + \cos(\xi_2-\xi_0)\cosh\alpha(\xi_2-\xi_0-\pi)\}\end{aligned} \tag{5.17}$$

for $\xi_0 < \xi_2 < 2\pi$.

From the second boundary condition (5.4*b*) we obtain, assuming that (5.11*a*) and (5.11*b*) are satisfied,

$$\int_0^\infty K_\alpha^{(1)}(s)\frac{\partial F}{\partial \xi_1}(\alpha,\xi_1+2\pi)\,d\alpha = \frac{\partial}{\partial \xi_1}\{[s-t_1]^{-\frac{1}{2}}-[s-\cos(\xi_1-\xi_0)]^{-\frac{1}{2}}\}, \quad (5.18)$$

$$\int_0^\infty K_\alpha^{(1)}(s)\frac{\partial F}{\partial \xi_2}(\alpha,\xi_2)\,d\alpha = \frac{\partial}{\partial \xi_2}\{[s-t_2]^{-\frac{1}{2}}-[s-\cos(\xi_2-\xi_0)]^{-\frac{1}{2}}\}. \quad (5.19)$$

If (5.14) and (5.15) are now inserted in (5.18)–(5.19), the differentiation and integration which occurs in each of the resulting equations may be interchanged, and we obtain finally

$$\frac{\partial F}{\partial \xi_1}(\alpha,\xi_1+2\pi) = \frac{\sqrt{2}}{\cosh\alpha\pi}\{\sin(\xi_0-\xi_1)\cosh\alpha[\pi-\xi_0+\xi_1]+\sin\xi_1\cosh\alpha(\xi_1-\pi)\} \quad (5.20)$$

for $0<\xi_1<\xi_0$, and

$$\frac{\partial F}{\partial \xi_2}(\alpha,\xi_2) = \frac{\sqrt{2}}{\cosh\alpha\pi}\{\sin\xi_2\cosh\alpha(\xi_2-\pi)-\sin(\xi_2-\xi_0)\cosh\alpha(\xi_2-\xi_0-\pi)\} \quad (5.21)$$

for $\xi_0<\xi_2<2\pi$.

If we let

$$\left.\begin{aligned} F_{11}(\alpha,\xi) &= \cos\xi\cosh\alpha\xi, & F_{12}(\alpha,\xi) &= \cos\xi\sinh\alpha\xi,\\ F_{21}(\alpha,\xi) &= \sin\xi\cosh\alpha\xi, & F_{22}(\alpha,\xi) &= \sin\xi\sinh\alpha\xi,\end{aligned}\right\} \quad (5.22)$$

and

$$g(\alpha,\xi) = \frac{-\sqrt{2}}{(\alpha^2+1)\cosh\alpha\pi}\left\{\begin{aligned} &\cos\xi\cosh\alpha(\xi-\pi)-\cos(\xi-\xi_0)\cosh\alpha(\xi-\xi_0+\pi)\\ &\quad-\alpha[\sin\xi\sinh\alpha(\xi-\pi)-\sin(\xi-\xi_0)\sinh\alpha(\xi-\xi_0+\pi)]\\ &\qquad\qquad (0<\xi<\xi_0),\\ &\cos\xi\cosh\alpha(\xi-\pi)-\cos(\xi-\xi_0)\cosh\alpha(\xi-\xi_0-\pi)\\ &\quad-\alpha[\sin\xi\sinh\alpha(\xi-\pi)-\sin(\xi-\xi_0)\sinh\alpha(\xi-\xi_0-\pi)]\\ &\qquad\qquad (\xi_0<\xi<2\pi),\end{aligned}\right\} \quad (5.23)$$

then (5.16) becomes, noting (5.7),

$$F_{11}(\alpha,\xi_1+2\pi)A(\alpha)+F_{12}(\alpha,\xi_1+2\pi)B(\alpha)+F_{21}(\alpha,\xi_1+2\pi)C(\alpha) + F_{22}(\alpha,\xi_1+2\pi)D(\alpha) = g(\alpha,\xi_1). \quad (5.24)$$

Further, (5.17), (5.20) and (5.21) yield 3 additional equations linear in $A,\ldots,D$. These with (5.24) constitute four equations which yield unique solutions for $A,\ldots,D$ by the use of Cramer's rule if the determinant

$$\mathscr{F}(\alpha,\xi_1,\xi_2) = \begin{vmatrix} F_{11}(\alpha,\xi_1+2\pi) & F_{12}(\alpha,\xi_1+2\pi) & F_{21}(\alpha,\xi_1+2\pi) & F_{22}(\alpha,\xi_1+2\pi)\\ F'_{11}(\alpha,\xi_1+2\pi) & F'_{12}(\alpha,\xi_1+2\pi) & F'_{21}(\alpha,\xi_1+2\pi) & F'_{22}(\alpha,\xi_1+2\pi)\\ F_{11}(\alpha,\xi_2) & F_{12}(\alpha,\xi_2) & F_{21}(\alpha,\xi_2) & F_{22}(\alpha,\xi_2)\\ F'_{11}(\alpha,\xi_2) & F'_{12}(\alpha,\xi_2) & F'_{21}(\alpha,\xi_2) & F'_{22}(\alpha,\xi_2)\end{vmatrix} \quad (5.25)$$

is non-vanishing. After a rather extensive calculation we find that

$$\mathscr{F}(\alpha,\xi_1,\xi_2) = \alpha^2\sin^2(\xi_2-\xi_1)-\sinh^2(\xi_2-\xi_1-2\pi)\alpha, \quad (5.26)$$

and it is easy to show that this is < 0 if $\alpha > 0$. For $\alpha = 0$, $\mathscr{F}$ does vanish, however, and Cramer's rule cannot be used, but we shall see later that this is a limiting case which causes no difficulty. The calculation of $A, \ldots, D$ is nothing more than a formidable exercise in algebraic manipulation, and will be omitted. We give merely the result of inserting these quantities in (5.7):

$$\begin{aligned}
&\frac{\mathscr{F}\cosh\alpha\pi}{\sqrt{2}}F(\alpha,\xi)\\
&\quad=\frac{1}{\alpha^2+1}\{\alpha^2\sin[\Lambda_1\sin(\xi_2-\xi)\cosh\alpha(\xi_1-\xi-2\pi)-\Lambda_2\sin(\xi_1-\xi)\cosh\alpha(\xi_2-\xi)]\\
&\qquad+\alpha[(\Lambda_1\cosh\alpha(\tau-\pi)-\Lambda_2\cos\tau)\sin(\xi_1-\xi)\sinh\alpha(\xi_2-\xi)\\
&\qquad+(\Lambda_2\cosh\alpha(\tau-2\pi)-\Lambda_1\cos\tau)\sin(\xi_2-\xi)\sinh\alpha(\xi_1-\xi+2\pi)]\\
&\qquad-\sinh\alpha(\tau-2\pi)[\cos(\xi_1-\xi)\sinh\alpha(\xi_2-\xi)-\cos(\xi_2-\xi)\sinh\alpha(\xi_1-\xi+2\pi)]\}\\
&\qquad-\alpha\sin\tau[\Gamma_1\sin(\xi_2-\xi)\sinh\alpha(\xi_1-\xi+2\pi)-\Gamma_2\sin(\xi_1-\xi)\sinh\alpha(\xi_2-\xi)]\\
&\qquad+\sinh\alpha(\tau-2\pi)[\Gamma_1\sin(\xi_1-\xi)\sinh\alpha(\xi_2-\xi)\\
&\qquad-\Gamma_2\sin(\xi_2-\xi)\sinh\alpha(\xi_1-\xi+2\pi)],
\end{aligned}\tag{5.27}$$

where $\tau = \xi_2 - \xi_1$, and

$$\Lambda_i = \frac{1}{\sqrt{2}}(\alpha^2+1)\,g(\alpha,\xi_i)\cosh\alpha\pi,\quad \Gamma_i = \frac{1}{\sqrt{2}}g'(\alpha,\xi_i)\cosh\alpha\pi\quad (i = 1, 2).\tag{5.28}$$

Here and elsewhere in this section the symbol $'$ indicates differentiation with respect to ξ.

As noted earlier, $A, \ldots, D$, and hence F, are not defined for $\alpha = 0$. Examination of (5.27) reveals, however, that $\lim_{\alpha\to 0} F(\alpha,\xi)$ is finite, so that the singularity of the integrand of (5.6) at $\alpha = 0$ is removable.

6. Particular lenticular configurations of interest

(a) The hemisphere

In this case $\xi_1 = \frac{1}{2}\pi$, $\xi_0 = \frac{3}{4}\pi$, $\xi_2 = \pi$, and we have

$$\begin{aligned}
&(\sinh^2\tfrac{3}{2}\pi\alpha-\alpha^2)\cosh\alpha\pi F(\alpha,\xi)\\
&\quad=\cos\xi\Bigg\{\left[\frac{\alpha M(\alpha)}{\alpha^2+1}\cosh\tfrac{3}{2}\pi\alpha-Q(\alpha)\sinh\tfrac{3}{2}\pi\alpha-\alpha\cosh\tfrac{3}{4}\pi\alpha\right]\\
&\qquad\times\sinh\alpha(\pi-\xi)-N(\alpha)\left[\frac{\alpha^2}{\alpha^2+1}\cosh\alpha(\pi-\xi)+\sinh\tfrac{3}{2}\pi\alpha\sinh\alpha(\tfrac{5}{2}\pi-\xi)\right]\Bigg\}\\
&\qquad+\sin\xi\Bigg\{\frac{M(\alpha)}{\alpha^2+1}[\sinh\tfrac{3}{2}\pi\alpha\sinh\alpha(\pi-\xi)+a^2\cosh\alpha(\tfrac{5}{2}\pi-\xi)]\\
&\qquad+\left[\frac{\alpha N(\alpha)}{\alpha^2+1}\cosh\tfrac{3}{2}\pi\alpha-\alpha Q(\alpha)-\cosh\tfrac{3}{4}\pi\alpha\sinh\tfrac{3}{2}\pi\alpha\right]\sinh\alpha(\tfrac{5}{2}\pi-\xi)\Bigg\},
\end{aligned}\tag{6.1}$$

where

$$\left.\begin{aligned}
M(\alpha) &= \cosh\tfrac{3}{4}\pi\alpha+\alpha(\sinh\tfrac{3}{4}\pi\alpha-\sqrt{2}\sinh\tfrac{1}{2}\pi\alpha),\\
N(\alpha) &= \sqrt{2}+\cosh\tfrac{3}{4}\pi\alpha+\alpha\sinh\tfrac{3}{4}\pi\alpha,\\
Q(\alpha) &= \cosh\tfrac{3}{4}\pi\alpha+\sqrt{2}\cosh\tfrac{1}{2}\pi\alpha.
\end{aligned}\right\}\tag{6.2}$$

It is hardly necessary to point out that much of the complexity (and little of the symmetry) of the general case is still present in this example. This is not surprising in view of the complex body shape considered.

(*b*) *The symmetrical (bi-convex) lens*

In this instance, if $\xi = \xi_1$ is one face of the lens, then the other is $\xi = \xi_2 = 2\pi - \xi_1$, and $\xi_0 = \pi$. We find that (5.28) and (5.23) with these values give

$$\begin{aligned}\Lambda_1 = \Lambda_2 &= -2\cosh\tfrac{1}{2}\alpha\pi[\cos\xi_1\cosh\alpha(\xi_1-\tfrac{1}{2}\pi) - \alpha\sin\xi_1\sinh\alpha(\xi_1-\tfrac{1}{2}\pi)],\\ \Gamma_1 = -\Gamma_2 &= 2\cosh\tfrac{1}{2}\alpha\pi\sin\xi_1\cosh\alpha(\xi_1-\tfrac{1}{2}\pi), \qquad (6.3)\end{aligned}$$

$$\mathscr{F}(\alpha,\xi_1,\xi_2) = \alpha^2\sin^2 2\xi_1 - \sinh^2 2\alpha\xi_1, \qquad (6.4)$$

and, after some manipulation, the specialization of (5.27) for this case can be written in the form

$$\begin{aligned}\frac{\mathscr{F}\cosh\alpha\pi}{2\sqrt{2}}F(\alpha,\xi) &= \cos\xi\cosh\alpha(2\pi-\xi)\left\{\frac{\Lambda_1}{\alpha^2+1}[\sinh\alpha\xi_1(-\cos\xi_1\sinh 2\alpha\xi_1\right.\\ &\quad \left. + \Phi\alpha\sin\xi_1) + \alpha^2\sin\xi_1\sin 2\xi_1\cosh\alpha\xi_1] - \Gamma_1\Psi\sin\xi_1\sinh\alpha\xi_1\right\}\\ &\quad + \sin\xi\sinh\alpha(2\pi-\xi)\left\{\frac{\Lambda_1}{\alpha^2+1}[\cosh\alpha\xi_1(\sin\xi_1\sinh 2\alpha\xi_1 + \Phi\alpha\cos\xi_1)\right.\\ &\quad \left. + \alpha^2\cos\xi_1\sin 2\xi_1\sinh\alpha\xi_1] - \Gamma_1\Psi\cos\xi_1\cosh\alpha\xi_1\right\}, \qquad (6.5)\end{aligned}$$

where
$$\left.\begin{aligned}\Phi(\alpha,\xi_1) &= \cos 2\xi_1 - \cosh 2\alpha\xi_1,\\ \Psi(\alpha,\xi_1) &= \alpha\sin 2\xi_1 - \sinh 2\alpha\xi_1.\end{aligned}\right\} \qquad (6.6)$$

Still further simplification can be achieved by noting that Ψ is a factor of $\mathscr{F}$ and of the coefficients of the Λ_1 terms. Removal of these common factors yields

$$\begin{aligned}&\frac{1}{2\sqrt{2}}\cosh\alpha\pi(\alpha\sin 2\xi_1 + \sinh 2\alpha\xi_1)F(\alpha,\xi)\\ &= \cos\xi\cosh\alpha(2\pi-\xi)\left\{\frac{\Lambda_1}{\alpha^2+1}(\alpha\sin\xi_1\cosh\alpha\xi_1 + \cos\xi_1\sinh\alpha\xi_1) - \Gamma_1\sin\xi_1\sinh\alpha\xi_1\right\}\\ &\quad + \sin\xi\sinh\alpha(2\pi-\xi)\left\{\frac{\Lambda_1}{\alpha^2+1}(\alpha\cos\xi_1\sinh\alpha\xi_1 - \sin\xi_1\cosh\alpha\xi_1) - \Gamma_1\cos\xi_1\cosh\alpha\xi_1\right\}.\end{aligned} \qquad (6.7)$$

A further specialization of interest is obtained by setting $\xi_1 = \frac{1}{2}\pi$, which gives the sphere. In this case $\Lambda_1 = \Lambda_2 = 0$, and $\Gamma_1 = -\Gamma_2 = 2\cosh(\frac{1}{2}\alpha\pi)$. It is easily checked that

$$F(\alpha,\xi) = -2\sqrt{2t}\,\frac{\cosh\alpha(2\pi-\xi)}{\cosh\alpha\pi}, \qquad (6.8)$$

and the integral in (5.6) then becomes

$$-2\sqrt{2t}\int_0^\infty \frac{K_\alpha^{(1)}(s)\cosh\alpha(2\pi-\xi)}{\cosh\alpha\pi}\,d\alpha. \qquad (6.9)$$

If we make use of the representation

$$K_\alpha(s) = \frac{\sqrt{2}}{\pi}\coth\alpha\pi\int_\eta^\infty \frac{\sin\alpha u\,du}{(\cosh u - s)^{\frac{1}{2}}} \qquad (6.10)$$

and the discussion of pp. 451–3 of Hobson (1931), it can be shown that

$$(s-t)^{-\frac{1}{2}} = \sqrt{2}\int_0^\infty \frac{K_\alpha(s)\cosh\alpha(\pi-\xi)\,d\alpha}{\cosh\alpha\pi} \quad (0 < \xi < 2\pi). \tag{6.11}$$

Differentiation of this and a permissible interchange with the integration on the right then yields

$$-\frac{1}{2\sqrt{2}}(s+t)^{-\frac{3}{2}} = \int_0^\infty \frac{K_\alpha^{(1)}(s)\cosh\alpha(2\pi-\xi)\,d\alpha}{\cosh\alpha\pi} \quad (\pi < \xi < 3\pi). \tag{6.12}$$

Thus (5.6) reduces to

$$\psi = \frac{Ur^2}{2}\left\{1-\left(\frac{s-t}{s+t}\right)^{\frac{1}{2}} - \frac{t(s-t)^{\frac{1}{2}}}{(s+t)^{\frac{3}{2}}}\right\}$$

and, since $(\rho/b)^2 = (s+t)/(s-t)$, we obtain

$$\psi = \frac{Ur^2}{2}\left(1-\frac{3}{2}\frac{b}{\rho}+\frac{1}{2}\frac{b^3}{\rho^3}\right) \tag{6.13}$$

which is the solution for the sphere given by Lamb (1945, p. 598).

(c) *The calotte, or spherical cap*

If $\xi_2 = \xi_1\ (= \xi_0)$, the two bounding surfaces of the lens coincide, and the body becomes a portion of a spherical surface bounded by a circle of latitude. This is called a calotte, or spherical cap. As ξ_0 goes from $\frac{1}{2}\pi$ to π the cap goes from a hemispherical surface of radius b to a flat disk of radius b. Cases $0 < \xi_0 < \frac{1}{2}\pi$ are of less interest. An approximation to the former type of body is found in the cup of the meteorologist's anemometer.

In this case the procedure used in obtaining the equation (5.27) for $F(\alpha,\xi)$ is invalid. Nevertheless, this formula, with $A, \ldots, D$ as determined there holds in the limit as $\xi_2 \to \xi_1$. To show this the expression $[s-t]^{-\frac{1}{2}} - [s-\cos(\xi-\xi_0)]^{-\frac{1}{2}}$ is represented by (5.15) for $\xi_0 < \xi < 2\pi$ and by (5.14) with ξ_1 replaced by $\xi_1 - 2\pi$ for $2\pi < \xi < 2\pi+\xi_0$. In either case the resulting integral is combined with that appearing in (5.6). The result is a form of ψ which is valid near the peak of the cap, i.e. in the neighbourhood of the point ($\xi = \xi_0$, $\eta = 0$) at which the x-axis pierces the cap.

We give the details of this process only for the case of the hemispherical cap. This is defined by $\xi_2 = \xi_1 = \xi_0 = \frac{1}{2}\pi$, and we have

$$g(\alpha,\xi_1) = g(\alpha,\xi_2) = \frac{\sqrt{2}}{(\alpha^2+1)\cosh\alpha\pi}[\cosh\alpha\pi - \alpha\sinh\tfrac{1}{2}\alpha\pi], \tag{6.14}$$

$$g'(\alpha,\xi_1) = g'(\alpha,\xi_2) = \frac{\sqrt{2}}{\cosh\alpha\pi}\cosh\tfrac{1}{2}\alpha\pi. \tag{6.15}$$

With the values of Λ_i and Γ_i which these give, (5.27) then yields

$$\begin{aligned} F(\alpha,\xi) = \frac{-\sqrt{2}}{(\alpha^2+1)\cosh^2\alpha\pi}\{&\alpha^2\cos\xi\cosh\alpha(\pi-\xi) \\ &+\alpha[\cosh\alpha\pi\cos\xi\sinh\alpha(\tfrac{3}{2}\pi-\xi)+\sinh\tfrac{1}{2}\alpha\pi\sin\xi\cosh\alpha(\tfrac{3}{2}\pi-\xi)] \\ &+\cosh\alpha(\tfrac{3}{2}\pi-\xi)[\cosh\tfrac{1}{2}\alpha\pi\cos\xi-\cosh\alpha\pi\sin\xi]\}. \end{aligned} \tag{6.16}$$

We may write (5.6) for $\xi_2 = \frac{1}{2}\pi$ in the form

$$\psi = \tfrac{1}{2}Ur^2(s-t)^{\frac{1}{2}}\left[(s-t)^{-\frac{1}{2}}-(s-\sin\xi)^{-\frac{1}{2}}-\int_0^\infty F(\alpha,\xi)\,K_\alpha^{(1)}(s)\,d\alpha\right] \quad (6.17)$$

and making use of (5.6) and (5.14), respectively, as indicated in the preceding paragraph,

$$\begin{aligned}(s-t)^{-\frac{1}{2}}&-(s-\sin\xi)^{-\frac{1}{2}}\\ &= -\sqrt{2}\int_0^\infty \frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi}\{\sin\xi[\alpha\sinh\alpha(\pi-\xi)+\cosh\alpha(\tfrac{3}{2}\pi-\xi)]\\ &\qquad +\cos\xi[\alpha\sinh\alpha(\tfrac{3}{2}\pi-\xi)+\cosh\alpha(\pi-\xi)]\}\,d\alpha \quad (\tfrac{1}{2}\pi<\xi<2\pi),\\ (s-t)^{-\frac{1}{2}}&-(s-\sin\xi)^{-\frac{1}{2}}\\ &= -\sqrt{2}\int_0^\infty \frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi}\{\sin\xi[\alpha\sinh\alpha(3\pi-\xi)-\cosh\alpha(\tfrac{3}{2}\pi-\xi)]\\ &\qquad +\cos\xi[\alpha\sinh\alpha(\tfrac{3}{2}\pi-\xi)+\cosh\alpha(3\pi-\xi)]\}\,d\alpha \quad (2\pi<\xi<\tfrac{5}{2}\pi). \quad (6.18)\end{aligned}$$

The insertion of these and (6.16) in (6.17) yields

$$\begin{aligned}\psi = \tfrac{1}{2}Ur^2(s-t)^{\frac{1}{2}}\int_0^\infty &\frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi}\{\alpha^2\cos\xi\cosh\alpha(\pi-\xi)+\sinh\alpha(\tfrac{1}{2}\pi-\xi)\\ &\times[\sinh\tfrac{1}{2}\alpha\pi\cos\xi-\alpha\cosh\tfrac{1}{2}\alpha\pi\sin\xi]\}\,d\alpha \quad (6.19a)\end{aligned}$$

for $\frac{1}{2}\pi < \xi < 2\pi$, and

$$\begin{aligned}\psi = \tfrac{1}{2}Ur^2(s-t)^{\frac{1}{2}}\int_0^\infty &\frac{K_\alpha^{(1)}(s)}{(\alpha^2+1)\cosh\alpha\pi}\{\alpha^2\cos\xi\cosh\alpha(\pi-\xi)\\ &-\sinh\alpha(\tfrac{5}{2}\pi-\xi)[\sinh\tfrac{3}{2}\alpha\pi\cos\xi+\alpha\cosh\tfrac{3}{2}\alpha\pi\sin\xi]\}\,d\alpha \quad (6.19b)\end{aligned}$$

for $2\pi < \xi < \frac{5}{2}\pi$.

Because of the formal nature of the operations yielding (6.19) it must be verified that the ψ thus defined is actually the solution of the cap problem. This is found to be the case. It is clear that ψ does not exhibit a singularity at the point where the x-axis pierces the cap.

7. The flow about a pair of separated spheres

The curves $\eta =$ const. defined by the transformation (5.1) constitute a family of circles in the xr-plane whose centres lie on the x-axis. All circles $\eta > 0$ lie entirely within $x > 0$ and enclose $(b, 0)$; those for which $\eta < 0$ lie entirely within $x < 0$ and enclose $(-b, 0)$. Rotation of these curves about the x-axis yields a family of spheres similarly characterized by $\eta \lessgtr 0$.

We consider two spheres defined by $\eta = \eta_1 > 0$ and $\eta = \eta_2 < 0$. It is clear that neither lies in the interior of the other and that they have no points in common; we refer to them as separated spheres (figure 4). The region exterior to both is defined by

$$\eta_2 < \eta < \eta_1 \quad (0 \leqslant \xi < 2\pi). \quad (7.1)$$

In order to solve the Stokes flow problem for the region (7.1) we seek solutions of (2.9) which are periodic in ξ of period 2π. We choose a decomposition for ψ_1 in (3.4) to be a sum of (a), (b) and (c) of (3.2), so that

$$\psi = \tfrac{1}{2}Ur^2[1-(\rho^2+b^2)\psi^3-x\psi^3]. \quad (7.2)$$

It is then easy to see that a suitable ψ is

$$\psi = \tfrac{1}{2}Ur^2\left\{1-(s-t)^{\frac{1}{2}}\sum_{n=1}^{\infty}[\bar{A}_r e^{n\eta}\cosh\eta+\bar{B}_n e^{-n\eta}\cosh\eta\right.$$
$$\left.+\bar{C}_n e^{n\eta}\sinh\eta+\bar{D}_n e^{-n\eta}\sinh\eta]\,P_n^{(1)}(\cos\xi)\right\} \quad (7.3)$$

which, by a redefinition of constants, can be written as

$$\psi = \tfrac{1}{2}Ur^2\left\{1-(s-t)^{\frac{1}{2}}\sum_{n=1}^{\infty}[A_n e^{(n+1)\eta}+B_n e^{(n-1)\eta}+C_n e^{-(n+1)\eta}\right.$$
$$\left.+D_n e^{-(n-1)\eta}]\,P_n^{(1)}(\cos\xi)\right\}. \quad (7.4)$$

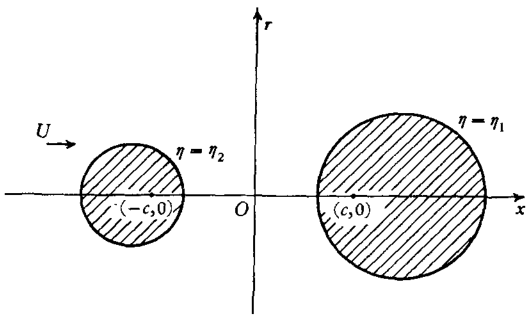

FIGURE 4. Separated spheres.

In order to satisfy the boundary conditions on $\eta = \eta_1$ and η_2 we recall the well-known expansion formulas (Hobson 1931, p. 335)

$$(s-t)^{-\frac{1}{2}} = \begin{cases} \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\eta}P_n(t) \quad (\eta>0), & (7.5a)\\ \sqrt{2}\sum_{n=0}^{\infty} e^{(n+\frac{1}{2})\eta}P_n(t) \quad (\eta<0). & (7.5b)\end{cases}$$

We now differentiate $(7.5a)$ with respect to t, multiply both sides of the result by $\sinh\eta$, and integrate with respect to η from $\eta_1\,(>0)$ to ∞. In this way we obtain

$$(s_1-t)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left\{\frac{e^{-(n+\frac{3}{2})\eta_1}}{n+\frac{3}{2}}-\frac{e^{-(n-\frac{1}{2})\eta_1}}{n-\frac{1}{2}}\right\}P_n^{(1)}(t) \quad (\eta_1>0) \qquad (7.6a)$$

and in a similar way find that

$$(s_2-t)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\sum_{n=1}^{\infty}\left\{\frac{e^{(n+\frac{3}{2})\eta_2}}{n+\frac{3}{2}}-\frac{e^{(n-\frac{1}{2})\eta_2}}{n-\frac{1}{2}}\right\}P_n^{(1)}(t) \quad (\eta_2<0). \qquad (7.6b)$$

With the aid of these results we now see by reference to (7.4) that the condition that ψ should vanish on $\eta=\eta_1$ and η_2 will be satisfied if we choose $A_n, \ldots, D_n$ to satisfy the relations

$$A_n e^{(n+1)\eta_i}+B_n e^{(n-1)\eta_i}+C_n e^{-(n+1)\eta_i}+D_n e^{-(n-1)\eta_i}$$
$$= \frac{\sqrt{2}}{2}\left[\frac{e^{-(n+\frac{3}{2})|\eta_i|}}{n+\frac{3}{2}}-\frac{e^{-(n-\frac{1}{2})|\eta_i|}}{n-\frac{1}{2}}\right] \quad (i=1,2). \qquad (7.7)$$

The second boundary condition, that $\partial\psi/\partial n = 0$ on $\eta = \eta_1$ and η_2, can be easily shown to reduce to

$$\left[\frac{\partial}{\partial\eta}(s-t)^{-\frac{1}{2}}\right]_{\eta=\eta_i} = \left[\frac{\partial}{\partial\eta}\sum_{n=1}^{\infty}\{A_n e^{(n+1)\eta}+B_n e^{(n-1)\eta}+C_n e^{-(n+1)\eta}+D_n e^{-(n-1)\eta}\}P_n^{(1)}(t)\right]_{\eta=\eta_i} \tag{7.8}$$

and hence will be satisfied if

$$(n+1)A_n e^{(n+1)\eta_i}+(n-1)B_n e^{(n-1)\eta_i}-(n+1)C_n e^{-(n+1)\eta_i}-(n-1)D_n e^{-(n-1)\eta_i}$$
$$= (-1)^i\tfrac{1}{2}\sqrt{2}\,[e^{-(n+\frac{3}{2})|\eta_i|}-e^{-(n-\frac{1}{2})|\eta_i|}] \quad (i=1,2). \tag{7.9}$$

Equations (7.7) and (7.9) constitute 4 equations for the determination of the constants $A_n, \ldots, D_n$, and hence the solution of the problem. We leave the problem at this point, since the solution has been given, although in somewhat different form, by Stimson & Jeffery (1926).

8. The flow about a spheroid

The solution of the Stokes flow problem for the prolate and oblate spheroids are implicit in the results of Oberbeck (1876) for the general ellipsoid. Since these problems furnish, however, striking examples of the ease with which problems can be solved by judicious choice of the representation formula for ψ, and since the results have a simpler form than that given by Oberbeck, the authors feel justified in sketching briefly the solution for these bodies by the methods of this paper.

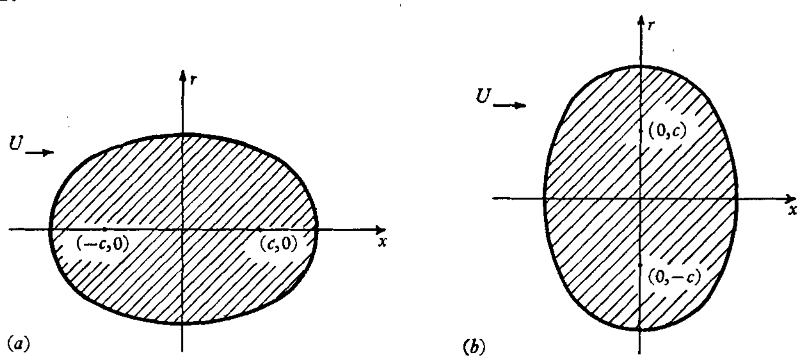

FIGURE 5. (*a*) Prolate spheroid. (*b*) Oblate spheroid.

The transformation
$$z = \cosh\zeta \quad (c > 0), \tag{8.1}$$
where $z = x+ir$ and $\zeta = \xi+i\eta$, serves to introduce elliptic co-ordinates in the xr-plane, line segments $\xi = \xi_0 = \text{const.}$, $0 < \eta < \pi$, of the plane being mapped into the upper halves ($r \geqslant 0$) of confocal ellipses in the xr-plane, with foci at $(\pm c, 0)$. Rotation of such a curve about the x-axis generates a prolate spheroid (figure 5*a*) whose exterior is defined by

$$\xi > \xi_0 \quad (0 \leqslant \eta \leqslant \pi). \tag{8.2}$$

We represent the solution of the differential equation $L_{-1}^2\psi = 0$ in the form

$$\psi = \tfrac{1}{2}Ur^2 - \psi_1 = \tfrac{1}{2}Ur^2(1-\psi^1-\psi^3). \tag{8.3}$$

It is easy to see (Hobson 1931, p. 413) that $Q_n(s)P_n(t)$ and $Q_n^{(1)}(s)P_n^{(1)}(t)$, $n = 0, 1, 2, \ldots$, where $s = \cosh\xi$, $t = \cos\eta$, are, respectively, functions ψ^1 and ψ^3 which are regular in the region (8.2). Thus, at least formally, we may choose

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - \sum_{n=0}^{\infty}[A_n P_n(t)\,Q_n(s) + B_n P_n^{(1)}(t)\,Q_n^{(1)}(s)]\right\}. \tag{8.4}$$

From the condition that $\partial\psi/\partial\eta$ vanish on $\xi = \xi_0$ it follows at once that we must have

$$A_1 = 0, \quad A_n, B_n = 0 \quad (n \geq 2),$$

and hence that

$$\psi = \tfrac{1}{2}Ur^2\{1 - A_0 P_0(t)\,Q_0(s) - B_1 P_1^{(1)}(t)\,Q_1^{(1)}(s)\}. \tag{8.5}$$

If we insert in this the expressions for $P_n(t)$ and $Q_n(s)$, $n = 0, 1$, and relabel the coefficients, the result is

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - A\ln\frac{s+1}{s-1} - B\frac{s}{s^2-1}\right\}. \tag{8.6}$$

The boundary conditions (3.3) now serve to determine A and B, and we find that

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - \frac{s(s_0^2-1)/(s_0^2+1) - \frac{1}{2}(s_0^2+1)\ln(s+1)/(s-1)}{s_0 - \frac{1}{2}(s_0^2+1)\ln(s_0+1)/(s_0-1)}\right\}, \tag{8.7}$$

where $s_0 = \cosh\xi_0$.

An easy calculation shows that the drag is

$$P = 8\pi\mu \lim_{\rho\to\infty}\frac{\rho\psi_1}{r_2} = 8\pi\,Uc\mu\left[\tfrac{1}{2}(s_0^2+1)\ln\frac{s_0+1}{s_0-1} - s_0\right]^{-1}. \tag{8.8}$$

The transformation

$$z = c\sinh\zeta \quad (c > 0), \tag{8.9}$$

yields oblate spheroids. The segments $\xi = \xi_0 = \text{const.}$, $0 < \eta < \pi$, of the ζ-plane are mapped into the halves of confocal ellipses with foci at $(0, \pm c)$ lying in $r \geq 0$. The rotation of such a curve about the x-axis generates an oblate spheroid (figure 5*b*) whose axis coincides with that of x. The exterior of the spheroid is given by

$$\xi > \xi_0 \quad (0 \leq \eta < \pi). \tag{8.10}$$

Again we represent the solution of (2.9) in the form

$$\psi = \tfrac{1}{2}Ur^2 - \psi_1 = \tfrac{1}{2}Ur^2(1 - \psi^1 - \psi^3). \tag{8.11}$$

In this case we find (Hobson 1931, p. 422) that $P_n(t)\,Q_n(i\tau)$ and $P_n^{(1)}(t)\,Q_n^{(1)}(i\tau)$, $n = 0, 1, 2, \ldots$, where $\tau = \sinh\xi$ and $t = \cos\eta$, are, respectively, functions ψ^1 and ψ^3 which are regular in (8.10). We may thus assume ψ to have the form

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - \sum_{n=0}^{\infty}[A_n Q_n(i\tau)\,P_n(t) + B_n Q_n^{(1)}(i\tau)\,P_n^{(1)}(t)]\right\}. \tag{8.12}$$

By a procedure similar to that used in the preceding case we find that the ψ which also satisfies the boundary conditions, and thus constitutes the solution of the problem is

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - \frac{\tau(1+\tau_0^2)/(1+\tau^2) + (1-\tau_0^2)\cot^{-1}\tau}{\tau_0 + (1-\tau_0^2)\cot^{-1}\tau_0}\right\}, \tag{8.13}$$

where $\tau_0 = \sinh\xi_0$.

The drag is found from (4.2) to be

$$P = 8\pi\mu Uc[\tau_0 + (1-\tau_0^2)\cot^{-1}\tau_0]^{-1}. \tag{8.14}$$

The case of the flat circular disk is of particular interest, and is the special case of (8.12) obtained by setting $\tau_0 = 0$. Accordingly,

$$\psi = \tfrac{1}{2}Ur^2\left\{1 - \frac{2}{\pi}\left(\cot^{-1}\tau + \frac{\tau}{1+\tau^2}\right)\right\} \tag{8.15}$$

and the drag is

$$P = 16\mu Uc. \tag{8.16}$$

It is not immediately obvious that (8.7) and (8.13) yield the same results as were obtained by Oberbeck, since he eschews the use of both the stream function and velocity potential, and deals directly with the velocity components, expressing them in terms of the gravitational potential of the ellipsoid and related integrals. For the ellipsoid of revolution, these integrals reduce to types which can be easily evaluated and the authors have done this to verify that in such case Oberbeck's formulas reduce to (8.8) and (8.14). The result (8.16) for the disk was also obtained by Oberbeck.

We note, finally, that for c fixed and ξ_0 large, both spheroids closely approximate a sphere of radius $\frac{1}{2}c\,e^{\xi_0}$, and it can be shown, as one would expect, that both (8.8) and (8.14) become approximately $6\pi\mu U(\frac{1}{2}c\,e^{\xi_0})$, the drag of a sphere of radius $\frac{1}{2}c\,e^{\xi_0}$.

9. The drag problem for the lens

The drag of the axially symmetric body is given by (4.2). In the case of the lens-shaped body ψ_1 consists of the last two terms of (5.6), where $F(\alpha, \xi)$ is obtained from (5.27)–(5.28). The evaluation of the formidable expression which results we will not attempt. The bi-convex lens, because of symmetry, gives an integral which is considerably simpler, but still requires several numerical integrations which the authors have not carried out. There is, however, one special case of interest in which the drag can be computed,† viz. that of the spherical cap. From (6.17) we find that

$$\psi_1 = \tfrac{1}{2}Ur^2\left[\left(\frac{s-t}{s-\sin\xi}\right)^{\frac{1}{2}} + (s-t)^{\frac{1}{2}}\int_0^\infty F(\alpha,\xi)\,K_\alpha^{(1)}(s)\,d\alpha\right] \tag{9.1}$$

and using (5.1) and (2.2) we have

$$\frac{P}{8\pi\mu} = \lim_{\substack{\xi\to 2\pi\\ \eta\to 0}} \frac{\rho\psi_1}{r^2} = \tfrac{1}{2}Ub\sqrt{2}\left[1 + \int_0^\infty F(\alpha, 2\pi)\,K_\alpha^{(1)}(1)\,d\alpha\right], \tag{9.2}$$

where

$$F(\alpha, 2\pi) = -\frac{\sqrt{2}}{(\alpha^2+1)\cosh^2\alpha\pi}\{\alpha^2\cosh\alpha\pi - \alpha\cosh\alpha\pi\sinh\tfrac{1}{2}\alpha\pi + \cosh^2\tfrac{1}{2}\alpha\pi\} \tag{9.3}$$

and

$$K_\alpha^{(1)}(1) = -\tfrac{1}{2}(\alpha^2 + \tfrac{1}{4}) \tag{9.4}$$

† Another such special case is the sphere, treated by Stokes. See §6 above, and the reference given there.

(Neumann 1881). Thus we find that

$$P = 4\sqrt{2}\pi\mu bU\left[1+\frac{\sqrt{2}}{8}\left\{4\int_0^\infty \alpha^2\operatorname{sech}\alpha\pi\,d\alpha - 4\int_0^\infty \frac{\alpha\sinh\frac{1}{2}\alpha\pi}{\cosh\alpha\pi}\,d\alpha - \int_0^\infty \operatorname{sech}\alpha\pi\,d\alpha\right.\right.$$
$$\left.\left.+2\int_0^\infty \operatorname{sech}^2\alpha\pi\,d\alpha + \frac{3}{2}\int_0^\infty \frac{\operatorname{sech}\alpha\pi}{\alpha^2+1}\,d\alpha + 3\int_0^\infty \frac{\alpha}{\alpha^2+1}\,\frac{\sinh\frac{1}{2}\alpha\pi}{\cosh\alpha\pi} - \frac{3}{2}\int_0^\infty \frac{\operatorname{sech}^2\alpha\pi}{\alpha^2+1}\,d\alpha\right\}\right]. \tag{9.5}$$

These integrals are all available in Bierens de Haan (1939), except for the last one. It can be evaluated in closed form by a contour integration and subsequent summation of a series which we will omit. We then obtain for the drag

$$P = 4\sqrt{2}\pi\mu bU\left[\frac{11}{8}+\frac{1}{\pi}-\frac{\sqrt{2}}{2}\right] = 17{\cdot}525Ub\mu.$$

This is somewhat in excess of the value $16Ub\mu$ which was obtained for the flat disk in the last section.

Body	Drag
Hemispherical cup	$4\sqrt{2}\pi\mu bU\left(\frac{11}{8}-\frac{\sqrt{2}}{2}+\frac{1}{\pi}\right) = 17{\cdot}525\mu bU$
Flat disk	$16\mu bU$
Sphere	$6\pi\mu bU$
Prolate spheroid	$8\pi\mu bU\left[\frac{1}{2}(s_0^2+1)\log\frac{s_0+1}{s_0-1}-s_0\right]^{-1}$
Oblate spheroid	$8\pi\mu bU[(1-\tau_0^2)\cot^{-1}\tau_0+\tau_0]^{-1}$
Two identical spheres	$8\pi\mu bU\sinh\alpha\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n-1)(2n+3)}\left\{1-\frac{4\sinh^2(n+\frac{1}{2})\alpha-(2n+1)^2\sinh^2\alpha}{2\sinh(2n+1)\alpha+(2n+1)\sinh\alpha}\right\}$

TABLE 1

Table 1 has been prepared to permit easy comparison of the cup drag with that of the other bodies which we have discussed. In every case b is the radius of the frontal area circle; s_0 and τ_0 are defined in §8, and $\alpha = \cosh^{-1}d$, where d is the ratio of the distance between centres of the circles to their diameter. The result for the two spheres is taken from Stimson & Jeffery (1926). It will be noted that the drag of the hemispherical cup lies about midway between that of the flat disk and the sphere. Calculation also shows that the drag of the oblate spheroid always lies between that of the disk and sphere, and the drag of the prolate spheroid is always greater than that of the sphere.

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